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On the asymptotic behavior of the solutions of a class of anisotropic initial-boundary value problems

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Abstract

In this paper, we investigate a class of anisotropic initial-boundary value problems, involving the Finsler-Laplacian. Firstly, by using a first-order differential inequality technique, we provide some appropriate conditions on the data which guarantee the blow-up of the solution at some explicit finite time. Next, under different appropriate conditions on the data, we will make use of a comparison principle to prove the global boundedness of the solution. Finally, by using a maximum principle for an appropriate P-function, in the sense of L.E. Payne, we derive some explicit exponential time decay bounds for the solution and its derivatives..

KEYWORDS

Finsler-Lapacian; anisotropic diffusion; blow-up; global boundedness; decay estimates

1 | INTRODUCTION

In this article, we investigate a class of anisotropic initial-value problems representing the mathematical model for the thermal conductivity in a material which conducts heat at different rates, depending on its crystallographic structure or composition. In most of the applications intensively studied so far, the heat diffusion is typically modeled as isotropic, which means that it is the same in all directions and the Laplace operator is the one which naturally appears into the corresponding heat diffusion equation. The mathematical model of the anisotropic heat conductivity requires consideration of the properties of the materials described above, so the Laplacian operator has to be replaced with a more complex operator and in this paper we consider the *Finsler-Laplacian*, which better describes the relation between the heat flux and the temperature gradient in each direction. Such problems are highly interesting for researchers from many fields of science and engineering, since they appear in many applications, such as, for instance, the design of electronic devices (microchips), aircrafts etc., where high heat fluxes can occur and the anisotropic diffusion of the heat can affect the performances of the devices, the failure or damage of the aircraft etc. Surprisingly, despite the importance of these problems in applications, it seems the literature is still very poor in results on this type of thermal conductivity. To our knowledge, at this moment there is only one paper in the literature discussing the anisotropic heat diffusion, involving the Finsler-Laplacian, namely the recent paper by G. Akagi, K. Ishige and R. Sato[1]. This paper is the first one dealing with some fundamental questions for the anisotropic parabolic problems described above, regarding the blow-up, global boundedness and time decay estimates of the solution.

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Let $n \geq 2$ and $H : \mathbb{R}^n \to [0, \infty)$ be a convex function of class $C^4(\mathbb{R}^n \setminus \{0\})$, which satisfies the following conditions

$$\begin{aligned} &H(t\xi) = |t|H(\xi) \text{ for any } t \in \mathbb{R},\ \xi \in \mathbb{R}^n,\\ &H \geq 0 \text{ in } \mathbb{R}^n \text{ and } H(\xi) = 0 \text{ if and only if } \xi = 0,\\ &Hess\left(H^2\right) \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{1.1}$$

We say that such a function is a norm on $\mathbb{R}^n$, since it has the properties of a norm. A typical example is the l_p-norm, that is

$$H(\xi) = \|\xi\|_p = \left(\sum_{i=1}^{n} \xi_i^p\right)^{1/p} \text{ for } p \in (1, \infty). \tag{1.2}$$

The *anisotropic Laplacian* (or *Finsler-Laplacian*) of a function $u : \mathbb{R}^n \to \mathbb{R}$, with respect to the above norm H, is defined by

$$\Delta_H u(x) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(H(\nabla u)H_{\xi_i}(\nabla u)\right). \tag{1.3}$$

where

$$H_{\xi_i} = \frac{\partial H}{\partial \xi_i}(\xi) \text{ and } \nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_n}(x)\right). \tag{1.4}$$

For instance, when H is the standard Euclidean norm, then the Finsler-Laplacian coincides with the standard Laplacian, that is $\Delta_H = \Delta$. Note also that here and in what follows we use $\xi \in \mathbb{R}^n$ for the argument of H and $x \in \mathbb{R}^n$ for the argumend of a function u.

The *Finsler-Laplace operator* has been intensively studied in the last few decades in various contexts, including both the Finsler geometry (see, for instance, M-Amar - G. Belletini[3], S.-I. Ohta[15], [16], S.-I. Ohta - K.-T. Sturm[17], [18], Z. Schen[25] and the references therein) and the partial differential equations, mainly of elliptic type (see, for instance, A. Alvino et al.[2], V. Ferone - B. Kawohl[8], A. Cianchi - P. Salani[6] and references therein).

This paper is concerned with the following class of initial-boundary value problems for the Finsler-Laplacian:

$$\begin{cases} \Delta_H u - u_t = f(u),\ x \in \Omega,\ \ t > 0,\\ u(x,t) = 0,\ x \in \partial\Omega,\ t > 0,\\ u(x,0) = g(x),\ x \in \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with a $C^{2,\epsilon}$-boundary $\partial\Omega$, while f and g are given functions assumed to satisfy the following conditions:

$$f \in C^1, \qquad f(0) = 0, \qquad f(x) \geq 0, \qquad s > 0, \tag{1.6}$$

$$g \in C^2, \qquad g \geq 0, \qquad g(x) = 0, \qquad x \in \partial\Omega. \tag{1.7}$$

Under these assumptions on the data, the maximum principle implies that $u(x,t)$ is nonnegative. Note also that since $Hess\left(H^2\right)$ is positive definite in $\mathbb{R}^n \setminus \{0\}$ and H^2 is homogeneous of degree 2, then H^2 is strictly convex in $\mathbb{R}^n$ and Δ_H is an uniformly elliptic operator in any compact subset of $\Omega \backslash \omega$, where $\omega := \{x \in \Omega : \nabla u = 0\}$. Since $H \in C^4(\mathbb{R}^n \setminus \{0\})$, we have by the classical regularity theory that the solution of (1.5) is of class C^3 on $\Omega \backslash \omega$ (see O. A. Ladyzenskaja-V. A. Solonnikov-N. N. Uralceva[12]). Therefore, we will consider as solutions of equation from (1.5) the strong solutions for which the equality in the equation holds almost everywhere.

It is well-known that the solutions of the initial-value problems may not exist for all time and that the only way that the solution can fail to exist is by becoming unbounded at some finite time t (see, for instance, the case $H(\xi) = |\xi|$ in J.M. Ball[4] or H. Kielhofer[11]). In Section 2 of this paper, we will establish conditions on the data of problem (1.5) forcing the solution $u(x,t)$ to blow up at some finite time t^* and, under these conditions, we derive an upper bound for the blow up time t^*. In Section 3, we determine conditions on the data sufficient to guarantee global boundedness of the solution. Finally, in Section 4, under some appropriate conditions on the data we will derive some explicit exponential decay estimates in time for the solution and its derivatives.

The results of this paper extend those obtained by L.E. Payne and G.A. Philippin[19], G.A. Philippin and V. Proytcheva[23], in the particular case $H(\xi) = |\xi|$. In order to handle our general case, our approach will follow naturally the techniques developed in these papers. Other similar investigations for various classes of parabolic problems have also been published in the last decade (see, for instance, L.E. Payne and G.A. Philippin[20],[21], L.E. Payne, G.A. Philippin and S. Vernier-Piro[22] or C. Enache[7]. For other results concerning the local and global existence of solutions of some partiular cases of problem (1.5) we refer the reader to the book of A.A. Samarskii and al.[24] or to the survey papers of H.A. Levine[13] and V.A. Galaktionov and J.L. Vazquez[10] and its references.

Finally, the notations $u_i := \frac{\partial u}{\partial x_i}, u_{ij} := \frac{\partial^2 u}{\partial x_i \partial x_j}, H_i := \frac{\partial H}{\partial \xi_i}, H_{ij} = \frac{\partial^2 H}{\partial \xi_i \partial \xi_j}$ will be used throughout this paper and summation from 1 to n is understood on repeated indices. Using these notations, for instance, we have

$$H_{ij}u_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 H}{\partial \xi_i \partial \xi_j}\frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{1.8}$$

2 | PRELIMINARIES. SOME USEFUL IDENTITIES AND INEQUALITIES

In this section, we remind some useful identities, inequalities and concepts that appear in the remaining part of this paper, when we prove of our main results.

Lemma 1. *(see Lemma 2.2 in L. Barbu - C. Enache[5]) If $H \in C^3(\mathbb{R}^N \setminus \{0\})$ is a positive homogeneous function of degree 1, then we have*

$$H_i(\xi)\xi_i = H(\xi), \quad H_{ij}(\xi)\xi_i = 0, \quad H_{ijk}(\xi)\xi_i = -H_{jk}(\xi), \tag{2.1}$$

for any $\xi \in \mathbb{R}^N \setminus \{0\}$ and $i, j, k \in \{1, \dots, N\}$.

Lemma 2. *(Kato inequality; see Lemma 2.2 in G. Wang - C. Xia[27]) At a point where $\nabla u \neq 0$, we have*

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq a_{ij}\, H_k H_l u_{ik}\, u_{jl}, \tag{2.2}$$

where $a_{ij}(\nabla u)(x) := \frac{\partial^2}{\partial \xi_i \partial \xi_j}(\frac{1}{2}H^2)(\nabla u(x))$.

Next, let us recall that there is another interesting function H^0, related to H, which is defined to be the support function of $K := \{x \in \mathbb{R}^n : H(x) < 1\}$, namely

$$H^0(x) := \sup_{\xi \in K} \langle x, \xi \rangle . \tag{2.3}$$

We call $W_H(x_0) := \left\{x \in \mathbb{R}^n : H^0(x - x_0) \leq r\right\}$ a *Wulff ball of radius r centered at* x_0. We say that $\gamma : [0, 1] \to \Omega$ is a *minimal geodesic from* x_1 *to* x_2 if

$$d_H(x_1, x_2) := \int_0^1 H^0\left(\dot{\gamma}(t)\right) dt = \inf \int_0^1 H^0\left(\dot{\overline{\gamma}}(t)\right) dt, \tag{2.4}$$

where the infimum is taken on all C^1 curves $\overline{\gamma}(t)$ in Ω from x_1 to x_2. In fact γ is a straight line and $d_H(x_1, x_2) = H^0(x_2 - x_1)$. We call d_H the *H-distance between* x_1 *and* x_2. Now we can define the *diameter* d_H of Ω with respect to the norm H on $\mathbb{R}^n$ as

$$d_H := \sup_{x_1, x_2 \in \overline{\Omega}} d_H(x_1, x_2). \tag{2.5}$$

In the same spirit we define the *inscribed inradius* i_H of Ω with respect to the norm H on $\mathbb{R}^n$ as the radius of the biggest Wulff ball that can be enclosed in $\overline{\Omega}$.

Finally, let us recall the definition of H-mean curvature of $S_t := \left\{x \in \overline{\Omega} : u = t\right\}$. To this end let ν be the outward normal of S_t, $\{e_\alpha\}_{\alpha=1}^{n-1}$ be a basis of the tangent space $T_p(S_t)$, $g_{\alpha\beta} = g\left(e_\alpha, e_\beta\right)$ be the first fundamental form, $\left(g^{\alpha\beta}\right)$ be the inverse matrix of $\left(g_{\alpha\beta}\right)$ and $\overline{\nabla}$ be the covariant derivative in $\mathbb{R}^n$. Then the *H-second fundamental form* $h^H_{\alpha\beta}$ and the *H-mean curvature* K_H are defined by

$$h^H_{\alpha\beta} := \left\langle H_{\xi\xi} \circ \overline{\nabla}_{e_\alpha}\nu, e_\beta \right\rangle, \qquad \text{and} \qquad K_H = g^{\alpha\beta}h^H_{\alpha\beta}. \tag{2.6}$$

We then have:

Lemma 3. *(see Lemma 2.4 in G. Wang - C. Xia[27]) Let u be a C^2 function with a regular level set $S_t := \{x \in \bar{\Omega} : \ u = t\}$. Let $K_H(S_t)$ be the H-mean curvature of the level set S_t. We then have*

$$\Delta_H u(x) = -HK_H(S_t) + H_iH_ju_{ij} = -HK_H(S_t) + \frac{\partial^2 u}{\partial \nu_H^2}, \tag{2.3}$$

for x with $u(x) = t$, *where* $\nu_H := H_\xi(\nu) = -H_\xi(\nabla u)$.

Finally, when $K_H \geq 0$, we say that $\partial\Omega$ is *H-mean convex*. In such a case we can also say that Ω is *H-convex*.

3 | SOME BLOW-UP CONDITIONS

In this section we will make use of a first order differential inequality technique, previously employed by G.A. Philippin and V. Proyetcheva[23] in the case $H(\xi) = |\xi|$ (see, also, C. Enache[7]), to establish some sufficient conditions on the data of problem (1.5) to produce blow-up of the solution $u(x,t)$ at some finite time t^*. Moreover, under these conditions, we will show that the same technique provides an explicit upper bound for t^*.

To this end, let us introduce the following auxiliary functions:

$$A(t) := \int_\Omega u^2(x,t)dx, \tag{3.1}$$

$$B(t) := \int_\Omega \left(F(u) - \frac{1}{2}H^2(\nabla u)\right) dx, \tag{3.2}$$

where

$$F(u) := \int_0^u f(s)\, ds. \tag{3.3}$$

The main result of this section is formulated in the following theorem:

Theorem 1. *Let* $u(x,t)$ *be the solution of the parabolic problem* (1.5) *and assume that the data of problem* (1.5) *satisfy the following conditions:*

$$sf(s) \geq \frac{1}{2}(4+\alpha)F(s), \quad s > 0, \tag{3.4}$$

where α *is a positive parameter, and*

$$B(0) = \int_\Omega \left(F(g) - \frac{1}{2}H^2(\nabla g)\right) dx \geq 0. \tag{3.5}$$

We then conclude that $u(x,t)$ *blows up at some finite time* $t^* < T$, *with*

$$T := \frac{4}{\alpha(\alpha+4)}A(0)B^{-1}(0) \leq \infty. \tag{3.6}$$

Proof. We first compute

$$\begin{aligned} A'(t) &= 2\int_\Omega uu_t\, dx = 2\int_\Omega u\left[\Delta_H u + f(u)\right] dx \\ &= -2\int_\Omega H^2(\nabla u)\, dx + 2\int_\Omega uf(u)dx \\ &\geq (4+\alpha)\, B(t), \end{aligned} \tag{3.7}$$

where we have successively used the differential equation (1.5), the divergence theorem, the assumption (3.4) and the definition of $B(t)$.

Next, we also compute

$$\begin{aligned} B'(t) &= \int_\Omega \left[fu_{,t} - HH_{\xi_i}u_{it}\right] dx = \int_\Omega \left[fu_t + (HH_{\xi_i})_i u_t\right] dx \\ &= \int_\Omega u_t\left[f(u) + \Delta_H u\right] dx = \int_\Omega u_t^2 dx \geq 0 \end{aligned} \tag{3.8}$$

where we have used again the divergence theorem. Therefore, $B(t)$ is a nondecreasing function in t, and we have

$$B(t) \geq B(0) \geq 0, \tag{3.9}$$

by (3.5).

Next, using the Schwarz inequality and inequalities (3.7) and (3.8), we obtain the following chain of inequalities:

$$AB' = \int_\Omega u^2 dx \int_\Omega u_t^2 dx \geq \left(\int_\Omega u u_t dx \right)^2 = \frac{1}{4}(A')^2 \geq \left(1 + \frac{\alpha}{4}\right) A'B. \tag{3.10}$$

Therefore,

$$\frac{d}{dt}(BA^{-(1+\frac{\alpha}{4})}) \geq 0. \tag{3.11}$$

So $BA^{-(1+\frac{\alpha}{4})}$ is a nondecreasing function of t and we have

$$BA^{-(1+\frac{\alpha}{4})} \geq B(0)A^{-(1+\frac{\alpha}{4})}(0) =: M \geq 0 \tag{3.12}$$

Next we integrate (3.11) and make use of (3.7) to obtain

$$-\frac{4}{\alpha(\alpha+4)}(A^{-\frac{\alpha}{4}})' = \frac{1}{\alpha+4}A'A^{-(1+\frac{\alpha}{4})} \geq BA^{-(1+\frac{\alpha}{4})} \geq M. \tag{3.13}$$

Integrating now (3.13) from 0 to t, we obtain the inequality

$$(A(t))^{-\alpha/4} \leq (A(0))^{-\alpha/4} - \frac{\alpha(\alpha+4)}{4}Mt, \tag{3.14}$$

which cannot hold for

$$t \geq T := \frac{4\alpha(\alpha+4)}{A(0)B^{-1}(0)}. \tag{3.15}$$

In conclusion, the solution $u(x,t)$ of problem (1.5) fails to exist by blowing up at some finite time $t^* < T$, with $T > 0$ given in (3.15). □

4 | SOME GLOBAL BOUNDEDNESS CONDITIONS

In this section, we will establish conditions on the data of problem (1.5) which prevent the blow up of $u(x,t)$ in finite time and guarantee the global boundedness of the solution. As in the particular case $H(\xi) = |\xi|$, investigated by G.A. Philippin and L.E. Payne in [19], we assume that the data of problem (1.5) satisfy the following condition

$$f(0) = 0, \qquad sf'(s) \geq f(s) > 0, \; s > 0. \tag{4.1}$$

We note that (4.1) implies that $f(s)/s$ is a non-decreasing function in s. The solution $u(x,t)$ of problem (1.5) can blow-up in a finite time t^*. In this case, the solution exists in an interval $(0,\tau)$, with $\tau < t^*$. We thus denote

$$u_m := \max_{\Omega\times(0,\tau)} u(x,t) < \infty. \tag{4.2}$$

Our aim is to determine some conditions on initial data $g(x)$ sufficient to guarantee that the blow-up does not occur. To this end, we make use of the first eigenvalue λ_1 of the Finsler-Laplacian and the corresponding function ϕ_1 for a region $\tilde{\Omega} \supseteq \Omega$:

$$\begin{cases} \Delta_H \phi_1(x) + \lambda_1 \phi_1(x) = 0, \; \phi_1(x) > 0, \; x \in \tilde{\Omega}, \\ \phi_1(x) = 0, \; \text{on } \partial\tilde{\Omega}. \end{cases} \tag{4.3}$$

Moreover, since $\phi_1(x)$ is determined up to an arbitrary multiplicative constant, we normalize $\phi_1(x)$ by the condition

$$\max_{\tilde{\Omega}} \phi_1(x) = 1. \tag{4.4}$$

The reason for replacing by $\tilde{\Omega} \supseteq \Omega$ in our investigation is merely to allow an explicit computation of ϕ_1 and λ_1, if possible, by considering some particular shapes for $\tilde{\Omega}$.

Lemma 4. *Let $u(x,t)$ be a solution of problem* (1.5), *where f satisfies* (4.1). *If τ is any time prior to blow-up time, then $u(x,t)$ satisfies the following estimate*

$$0 \le u(x,t) \le \Gamma_1 exp\left(-\left(\lambda_1 - \frac{f(u_m)}{u_m}\right)t\right), \quad t \in [0,\tau], \tag{4.5}$$

where

$$\Gamma_1 := \max_{\Omega}\left(\frac{g(x)}{\phi_1(x)}\right). \tag{4.6}$$

Proof. First inequality in (4.5) follows from the maximum principle. To obtain the second inequality, we consider the following auxiliary function

$$v(x,t) = u(x,t)exp\left(-\frac{f(u_m)}{u_m}t\right), \tag{4.7}$$

and compute

$$\begin{aligned}(\Delta_H v - v_t)\, exp\left(-\tfrac{f(u_m)}{u_m}t\right) &= \Delta_H u - u_t + \tfrac{f(u_m)}{u_m}u \\ &\ge \Delta_H u - u_t + f(u) = 0.\end{aligned} \tag{4.8}$$

We then have

$$\begin{cases}\Delta_H v - v_t \ge 0, \ x \in \Omega, \ t \in (0,\tau), \\ v(x,0) = g(x), \ x \in \Omega, \\ v(x,t) = 0, \ x \in \partial\Omega, \ t \in (0,\tau).\end{cases} \tag{4.9}$$

The comparison principle then implies that

$$v(x,t) \le w(x,t) := \Gamma_1 \phi_1 e^{-\lambda_1 t}, \tag{4.10}$$

because we have

$$\begin{cases}\Delta_H w - w_t = 0, \ x \in \Omega, \ t \in (0,\tau). \\ w(x,0) = \Gamma_1\phi_1(x) \ge g(x), \ x \in \Omega, \\ w(x,t) \ge 0, \ x \in \partial\Omega, \ t \in (0,\tau).\end{cases} \tag{4.11}$$

Now, the combination of (4.7) and (4.10) implies the desired inequality (4.5). □

Theorem 2. *Under the assumptions of the previous lemma, if Γ_1 also satisfies the condition*

$$\frac{f(\Gamma_1)}{\Gamma_1} < \lambda_1, \tag{4.12}$$

then $t^* = \infty$ and we have

$$\max_{\Omega}\frac{f(u(x,t))}{u(x,t)} < \lambda_1, \ 0 \le t < \infty. \tag{4.13}$$

Proof. We suppose that (4.13) is not true and establish a contradiction. By continuity, there exists a first time $\tilde{t}$ for which $f(u)/u$ reaches the value λ_1, that is

$$\max_{\Omega}\frac{f(u(x,\tilde{t}))}{u(x,\tilde{t})} = \lambda_1. \tag{4.14}$$

Since $f(s)/s$ is a nondecreasing function of $s > 0$, Lemma 4 implies

$$u(x,t) \le \Gamma_1, \ 0 \le t \le \tilde{t}, \tag{4.15}$$

which leads to the following chain of inequalities

$$\frac{f(u(x,t))}{u(x,t)} \le \frac{f(\Gamma_1)}{\Gamma_1} < \lambda_1, \ x \in \Omega, \ 0 \le \tilde{t}, \tag{4.16}$$

in view of (4.12). In particular, we have

$$\max_{\Omega}\frac{f(u(x,\tilde{t}))}{u(x,\tilde{t})} < \lambda_1, \tag{4.17}$$

which is in contradiction with the definition of $\tilde{t}$. We then conclude that $\tilde{t} = \infty$ and the proof of the theorem is complete. □

5 | SOME EXPLICIT TIME DECAY ESTIMATES

In this section, we will establish sufficient conditions on the data to derive some exponential time decay bounds for $u(x,t)$, the solution of problem (1.5). For this aim, we shall derive some maximum principle for an appropriate combination of u, H and data of problem (1.5). The combination that we consider will be of the following form (see G. Wang - C. Xia[26] or L. Barbu - C. Enache[5] for the stationary case):

$$P(x,t) = \left\{ H^2(\nabla u)\, dx + 2\int_0^u f(s)ds + \alpha u^2 \right\} e^{2\alpha t}, \tag{5.1}$$

where α is a real positive parameter to be appropriately chosen.

The main result of this section is formulated in the following theorem:

Theorem 3. *Let $u(x,t)$ be the solution of problem* (1.5)*. Assume that Ω is H-convex and condition* (4.1) *holds. Assume also that Ω and $g(x) \geq 0$ are small enough in the following sense*

$$\frac{f(\Gamma_1)}{\Gamma_1} \leq \frac{\pi^2}{4i_H^2} - \alpha. \tag{5.2}$$

Then the auxiliary function $P(x,t)$, defined in (5.1)*, takes its maximum value at $t = 0$, so that we have*

$$H^2(\nabla u) + 2\int_0^u f(s)ds + \alpha u^2 \leq M^2 e^{-2\alpha t}, \tag{5.3}$$

where

$$M^2 := \max_{\Omega} \left\{ H^2(\nabla g) + \alpha \int_0^g f(s)ds + \alpha g^2 \right\}. \tag{5.4}$$

Proof. The proof of the theorem is given in several steps.

Step 1. First of all, let us remind that the solution of (1.5) is of class C^3 on $\Omega\backslash\omega$, so that we can differentiating (5.1), to obtain successively in $\Omega\backslash\omega$ that

$$P_i = 2\left\{HH_k u_{ki} + f u_i + \alpha u u_i\right\} e^{2\alpha t}, \tag{5.5}$$

$$\begin{aligned} P_{ij} &= 2\left\{H_l u_{lj} H_k u_{ki} + HH_{kl} u_{lj} u_{ki} + HH_k u_{kij}\right. \\ &\left. + f' u_i u_j + f u_{ij} + \alpha u_j u_i + \alpha u u_{ij}\right\} e^{2\alpha t}, \end{aligned} \tag{5.6}$$

respectively

$$P_t = 2\left\{HH_k u_{kt} + f u_t + \alpha u u_t + \alpha H^2 + 2\alpha \int_0^u f(s)ds + \alpha^2 u^2\right\} e^{2\alpha t}. \tag{5.7}$$

Moreover, from (5.5) we can derive the following useful identities :

$$H_k u_{ki} = \frac{P_i e^{-2\alpha t}}{2H} - \frac{f}{H} u_i - \alpha \frac{u}{H} u_i, \tag{5.8}$$

and

$$H_i H_k u_{ki} = \frac{H_i P_i e^{-2\alpha t}}{2H} - \frac{H_i f}{H} u_i - \alpha H_i \frac{u}{H} u_i = \frac{H_i P_i e^{-2\alpha t}}{2H} - f - \alpha u. \tag{5.9}$$

Let us now remind that

$$a_{ij}(\nabla u)(x) := \frac{\partial^2}{\partial\xi_i \partial\xi_j}\left(\frac{1}{2}H^2\right)(\nabla u(x)) = (H_i H_j + HH_{ij})(\nabla u(x)). \tag{5.10}$$

Then, we can rewrite the equation (1.5) as

$$a_{ij} u_{ij} = (H_i H_j + HH_{ij}) u_{ij} = u_t - f. \tag{5.11}$$

Moreover, making use of (5.8) in (5.11), we get

$$\begin{aligned} HH_{ij} u_{ij} &= -\frac{H_i P_i e^{-2\alpha t}}{2H} + \frac{f}{H} H_i u_i + \alpha \frac{u}{H} H_i u_i + u_t - f \\ &= -\frac{H_i P_i e^{-2\alpha t}}{2H} + \alpha u + u_t. \end{aligned} \tag{5.12}$$

On the other hand, differentiating (5.11) we have

$$H_l H_{ij} u_{lk} u_{ij} + H H_{ijl} u_{lk} u_{ij} + 2 H_{il} H_j u_{lk} u_{ij} + a_{ij} u_{ijk} = u_{tk} - f' u_k \tag{5.13}$$

But by Lemma 2.1 we have

$$H_{ij} u_j = 0 \text{ for all } i. \tag{5.14}$$

Then by taking derivative of (5.14) with respect to x_i and summing over i, we get

$$H_{ij} u_{ij} + H_{ijl} u_{li} u_j = 0. \tag{5.15}$$

Next, we compute

$$\begin{aligned} a_{ij} P_{ij} - P_t &= 2 a_{ij} \left\{ H_l u_{lj} H_k H_{ki} + H H_{kl} u_{lj} u_{ki} + H H_k u_{kij} + f' u_i u_j + f u_{ij} \right. \\ &\quad \left. + \alpha u_i u_j + \alpha u u_{ij} \right\} e^{2\alpha t} - 2 \left\{ H H_k u_{kt} + f u_t + \alpha u t \right. \\ &\quad \left. + \alpha H^2 + 2\alpha F(u) + \alpha^2 + u^2 \right\} e^{2\alpha t} \text{ in } \Omega \backslash \omega. \end{aligned} \tag{5.16}$$

Making now use of (5.10) and (5.11) in (5.16), after some simplifications we get

$$\begin{aligned} a_{ij} P_{ij} - P_t &= 2 \left\{ H_i H_j H_l u_{lj} H_k u_{ki} + H^2 H_{ij} H_{kl} u_{lj} u_{ki} - H H_k H_l H_{ij} u_{lk} u_{ij} \right. \\ &\quad - H^2 H_k H_{ijl} u_{lk} u_{ij} + f' H^2 + f'(H H_{ij} + H_i H_j) u_i u_j - f^2 \\ &\quad + \alpha (H H_{ij} + H_i H_j) u_i u_j - \alpha u f - H H_k u_{kt} \\ &\quad \left. - \alpha H^2 - 2\alpha F - \alpha^2 u^2 \right\} e^{2\alpha t} \text{ in } \Omega \backslash \omega. \end{aligned} \tag{5.17}$$

Next, we compute separately some of the above terms. First, using (5.9) we get

$$H_i H_j H_l H_k u_{lj} u_{ki} = \left(\frac{H_i P_i e^{-2\alpha t}}{2H} - f - \alpha u \right)^2 = (f + \alpha u)^2 + \ldots \tag{5.18}$$

where here and in what follows the dots stand for terms in P_k.

On the other hand, using (5.10) we have

$$\begin{aligned} H^2 H_{ij} H_{lk} u_{lj} u_{ki} &= (a_{ij} - H_i H_j) u_{jl} (a_{kl} - H_k H_l) u_{ki} \\ &= a_{ij} u_{jl} a_{kl} u_{ki} + H_i H_j u_{jl} H_k H_l u_{ki} - 2 H_i H_j a_{kl} u_{jl} u_{ki} \\ &= a_{ij} u_{jl} a_{kl} u_{ki} + H_i H_j u_{jl} H_k H_l u_{ki} - 2 H_i H_j a_{kl} u_{jl} u_{ki} \\ &= a_{ij} u_{jl} a_{kl} u_{ki} + \left[\tfrac{H_i P_i}{2H} e^{-2\alpha t} - (f + \alpha u) \right]^2 \\ &\quad + 2 \left[\tfrac{P_l}{2H} e^{-2\alpha t} - \tfrac{f + \alpha u}{H} u_l \right] \left[\tfrac{P_k}{2H} e^{-2\alpha t} - \tfrac{f + \alpha u}{H} u_k \right] \left[H_k H_l + H H_{ll} \right]. \end{aligned} \tag{5.19}$$

Now using Kato inequality (see Lemma 2) and (5.8), we get

$$\begin{aligned} a_{ij} u_{jl} a_{kl} u_k &\geq a_{ij} H_k H_l u_{lk} u_{jl} = a_{ij} \left(\tfrac{P_i}{2H} e^{-2\alpha t} - \tfrac{f + \alpha u}{H} u_i \right) \left(\tfrac{P_j}{2H} e^{-2\alpha t} - \tfrac{f + \alpha u}{H} u_j \right) \\ &= a_{ij} \tfrac{(f + \alpha u)^2}{H^2} u_i u_j + \ldots = (\underbrace{H_i H_j u_i u_j}_{=H^2} + \underbrace{H H_{ij} u_i u_j}_{=0}) \tfrac{(f + \alpha u)^2}{H^2} + \ldots \\ &= (f + \alpha u)^2 + \ldots. \end{aligned} \tag{5.20}$$

Replacing (5.20) into (5.19) and opening up the parentheses, we get

$$\begin{aligned} H^2 H_{ij} H_{lk} u_{lj} u_{ki} &\geq 2(f + \alpha u)^2 - 2 \tfrac{(f + \alpha u)^2}{H^2} \underbrace{H_k u_k H_l u_l}_{=H^2} - 2 \tfrac{(f + \alpha u)^2}{H^2} \underbrace{H H_{kl} u_k u_l}_{=0} + \ldots \\ &\geq 0 + \ldots. \end{aligned} \tag{5.21}$$

Next, we use again (5.9) to compute

$$\begin{aligned} HH_kH_lH_{ij}u_{lk}u_{ij} &= \left[\tfrac{H_lP_l}{2H}e^{-2\alpha t} - (f+\alpha u)\right]\left[-\tfrac{H_iP_i}{2H}e^{-2\alpha t} + \alpha u + u_t\right] \\ &= -(f+\alpha u)(\alpha u + u_t) + \ldots. \end{aligned} \tag{5.22}$$

Finally, using again (5.8) and (5.12), we get

$$\begin{aligned} H^2H_kH_{ijl}u_{lk}u_{ij} &= H^2H_{ijl}\left[\tfrac{P_l}{2H}e^{-2\alpha t} - \tfrac{(f+\alpha u)u_l}{H}\right]u_{ij} = H^2H_{ijl}\tfrac{(f+\alpha u)}{H}u_lu_{ij} + \ldots \\ &= HH_{ij}u_{ij}(f+\alpha u) + \ldots \\ &= \left[\tfrac{H_iP_i}{2H}e^{-2\alpha t} + (\alpha u + u_t)\right](f+\alpha u) + \ldots \\ &= (\alpha u + u_t)(f+\alpha u) + \ldots \end{aligned} \tag{5.23}$$

Substituting now (5.18), (5.21), (5.22) and (5.23) into (5.17), after simplifications we obtain

$$a_{ij}P_{ij} - P_t + \ldots \geq 2\alpha\left[uf - 2F\right]e^{2\alpha t} \geq 0, \text{ in } \Omega\backslash\omega. \tag{5.24}$$

It then follows from Nirenberg's type maximum principle[14] that P takes its maximum value either:

(i) at a point $\mathbf{P}$ on $\partial\Omega$ for some $t > 0$; or

(ii) at a critical point of $u(x,t)$ for some $t > 0$; or

(iii) at a point $\mathbf{P}$ in Ω at time $t = 0$.

In the next two steps, we will show that under our assumptions (i) and (ii) can not hold.

Step 2. In what follows we'll use Friedman's type maximum principle[9] to show that $P(x,t)$ cannot take its maximum value on $\partial\Omega$, so possibiity (i) is eliminated.

Indeed, suppose that $P(x,t)$ takes its maximum value at $\hat{P} = (\hat{x},\hat{t})$ on $\partial\Omega$. Then by Friedman's type maximum principle, we have

$$\begin{aligned} \frac{\partial P}{\partial\nu_H}\frac{\partial P}{\partial\nu_H} &= 2\left[HH_iu_{ij}\nu_H^j + f\frac{\partial u}{\partial\nu_H} + \alpha u\frac{\partial u}{\partial\nu_H}\right]e^{2\alpha t} \\ &= 2\left[HH_iu_{ij}\nu_H^j\right]e^{2\alpha t} > 0. \end{aligned} \tag{5.25}$$

On the other hand, from differential equation (1.5) evaluated on $\partial\Omega \in C^{2,\epsilon}$, we have $\Delta u = 0$, or equivalently

$$-HK_H + \frac{\partial^2 u}{\partial\nu_H^2} = 0. \tag{5.26}$$

Moreover, since

$$HH_iu_{ij}\nu_H^j = -\frac{\partial^2 u}{\partial\nu_H^2}, \tag{5.27}$$

we get

$$HH_iu_{ij}\nu_H^j = -HK_H, \tag{5.28}$$

so that (5.25) implies the following

$$-HK_H > 0, \tag{5.29}$$

which contradicts the fact that $K_H \geq 0$. Note that $\nabla u \neq 0$ on $\partial\Omega$ in view of Friedman's maximum principle, so $H(\nabla u) \neq 0$ on $\partial\Omega$.

Step 3. We suppose that the second possibility (ii) holds, i.e. $P(x,t)$ takes its maximum at a critical point $\bar{Q} = (\bar{x},\bar{t})$. Then we have

$$P(x,t) \leq P(\bar{x},\bar{t}),\ x \in \Omega,\ t > 0. \tag{5.30}$$

Evaluating (5.30) at $t = \bar{t}$, we get

$$H^2(\nabla u) \leq 2\int_u^{u_m} f(s)ds + \alpha(u_m^2 - u^2), \tag{5.31}$$

where $u_m = \max_{\Omega} u(x,\bar{t})$. Using now Cauchy's mean value theorem we can write

$$\begin{aligned} 2\int_u^{u_m} f(s)ds &= 2\left[\int_0^{u_m} f(s)ds - \int_0^{u} f(s)ds\right] = \frac{f(\xi)}{\xi}(u_m^2 - u^2(x,\bar{t})) \\ &\leq \frac{f(u_m)}{u_m}(u_m^2 - u^2(x,\bar{t})), \end{aligned} \tag{5.32}$$

where ξ is an intermediate value between u and u_m and in the last step we used the fact that $f(u)/u$ is monotone increasing. The insertion of (5.32) into (5.31) leads to the following inequality

$$H^2(\nabla u) \leq \left[\alpha + \frac{f(u_m)}{u_m}\right](u_m^2 - u^2). \tag{5.33}$$

We are now choosing a point $\hat{x} \in \partial\Omega$ with

$$d_H(\bar{x},\hat{x}) = d_H(\bar{x},\partial\Omega) \leq i_H, \tag{5.34}$$

and $\gamma(t) : [0,1] \to \bar{\Omega}$ the minimal geodesic connecting $\bar{x}$ with $\hat{x}$. Using the estimate (5.33) and integrating along the geodesic connecting $\bar{x}$ and $\hat{x}$ we obtain

$$\begin{aligned} \frac{\pi}{2} &= \int_0^{u_m} \frac{1}{\sqrt{u_m^2 - u^2}} du \leq \sqrt{\alpha + \frac{f(u_m)}{u_m}} \int_0^1 \frac{1}{H(\nabla u)} du \\ &= \sqrt{\alpha + \frac{f(u_m)}{u_m}} \int_0^1 \frac{\langle \nabla u(\gamma(t)), \dot{\gamma}(t)\rangle}{H(\nabla u(\gamma(t)))} dt \leq \sqrt{\alpha + \frac{f(u_m)}{u_m}} \int_0^1 H^o(\dot{\gamma}(t))dt \\ &\leq \sqrt{\alpha + \frac{f(u_m)}{u_m}}\, i_H, \end{aligned} \tag{5.35}$$

where the Cauchy-Schwarz inequality was used in the last step. Therefore,

$$\frac{\pi^2}{4i_H^2} \leq \alpha + \frac{f(u_m)}{u_m}. \tag{5.36}$$

The inequality (5.36) is a necessary condition in order that $P(x,t)$ takes its maximum at a critical point of $u(x,t)$. On the other hand using (5.2) and the fact that $f(s)/s$ is a nondecreasing function we obtain the following chain of inequalities,

$$\frac{f(u_m)}{u_m} \leq \frac{f(\Gamma_1)}{\Gamma_1} < \frac{\pi^2}{4i_H^2} - \alpha, \tag{5.37}$$

which is in contradiction with (5.35). This completes the proof of the theorem. □

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