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Generalized Filtered Lifting Line Theory for Arbitrary Chord Lengths and Application to Wind Turbine Blades

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Abstract

The filtered lifting line theory is an analytical approach to solving the equations of flow subjected to body forces with a Gaussian distribution, such as used in the actuator line model. In the original formulation[1], the changes in chord length along the blade were assumed to be small. This assumption can lead to errors in the induced velocities predicted by the theory compared to full solutions of the equations. In this work, we revisit the original derivation and provide a more general formulation, that can account for significant changes in chord along the blade. The revised formulation allows for applications to wings with significant changes in chord along the span, such as wind turbine blades.

KEYWORDS:
aerodynamics, lifting line, actuator line model

1 | INTRODUCTION

The filtered lifting line theory was developed to obtain solutions of flow subjected to Gaussian body forces, as used in the actuator line model formulation[1,2]. The original formulation assumed that the changes in chord along the blade were small, making it possible to express the induced velocity analytically as an integral convolution equation. However, this assumption does not hold for many wings (specifically wind turbine blades). In this note, we propose a generalization of the original formulation to take into account strong changes in chord (and Gaussian smoothing kernel ϵ) along the blade. The generalized theory is also analytical and preserves the usefulness of the filtered lifting line theory and its applicability to the actuator line model. Also, we apply the approach in simulation of a fixed wing with a chord distribution similar to a wind turbine blade. We compare the results using filtered lifting line theory correction to solutions from large eddy simulations (LES) using an actuator line. We highlight the differences between corrected and uncorrected cases, implement the new formulation, rerun simulations, and highlight the improvements when using the generalized formulation.

2 | FILTERED LIFTING LINE THEORY CORRECTION

The filtered lifting line theory expresses the induced velocity along a wing, $u'_y(z)$, as the solution to the integro-differential equation[1]:

$$u'_y(z) = -\int_0^S \frac{dG(z')}{dz'} \frac{\left(1 - e^{-(z-z')^2/\epsilon(z')^2}\right)}{U_\infty(z')\, 4\pi(z-z')}\, dz' \tag{1}$$

where $G(z') = \frac{1}{2} c_L(\alpha) c W^2$ is the lift force per unit span, U_∞ is the inflow velocity, $W = \sqrt{U_\infty^2 + {u'_y}^2}$ is the magnitude of the velocity vector, S is the wing span, c_L is the lift coefficient, α is the angle of attack that depends on $u'_y(z')$, c is the chord, ϵ is the width of the body force kernel and/or vortex core size, and z is the spanwise coordinate.

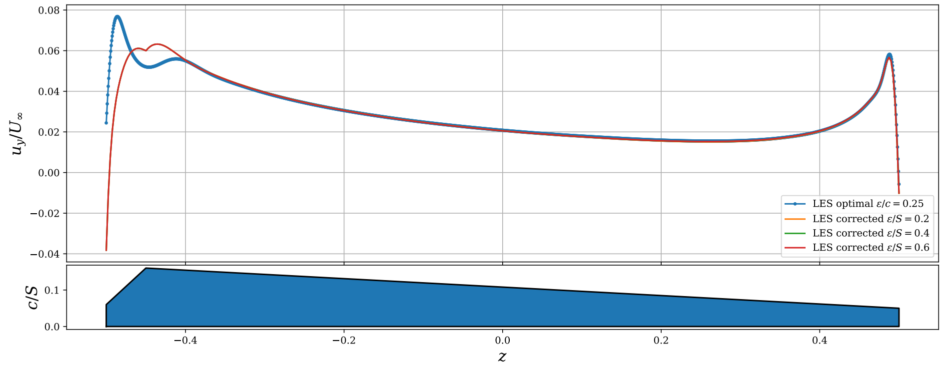

Figure 1 The induced velocity from the LES using the filtered lifting line theory correction do not match the results with optimal ϵ without the correction. The chord profile of the wing is shown below.

The width of the forces distributed in the filtered lifting line formulations is given by ϵ. Previous work suggest that a value of ϵ on the order of 0.25c provides velocity distributions arising from the Gaussian forcing that most closely resemble those of ideal flow over a lifting airfoil with chord length c [1,3]. Equation 1 is used then to correct coarse-grained simulations, typically LES, that cannot afford the grid resolutions required to run with the optimal width ($\epsilon_{\text{opt}} = 0.25c$). The correction is done by analytically computing the difference between the induced velocity in the LES (from using ϵ_{LES}) and the desired optimal (ϵ_{opt}) and adding the difference to the velocity sampled from the LES. This approach has recently been used to reduce the dependency on grid resolution and ϵ from simulations of fixed wings and wind turbines blades using the actuator line model [1,4,5,6,7,8].

3 | SHORTCOMINGS OF THE THEORY

We now focus on a case where the correction has significant shortcomings. We perform LES of a wing that has a chord distribution similar to a wind turbine blade using the actuator line model. We use AMR-Wind, an incompressible LES code meant for wind energy applications using structured grids and adaptive mesh refinement [9,10]. We perform simulations of a fixed wing in uniform inflow with a baseline angle of attack $\alpha = 6^o$. We use a large domain (16S in all directions) with grid refinements to ensure that the effect on induced velocities from domain size is less than 0.1%. The grid spacing (and levels of refinement) is set to always meet the criteria of $\epsilon/\Delta > 4$ [11]. Our tests indicated that there is no difference in the induced velocities by further increasing the resolution. The simulations use 400 actuator points along the blade, and ϵ/S =0.1, 0.2, 0.4, 0.6 (typical values used in wind turbine simulations using the actuator line model).

Figure 1 shows the induced velocity from LES using the filtered lifting line theory correction compared to a simulation with fine resolution (symbols) and without the correction. The cases with the correction all collapse onto each other, as expected based on the original formulation [1]. However, there is disagreement near the root of the blade, where the changes in chord occur over distances comparable to c. Careful examination led us to understand that the difference is caused by the theory's limitation to $d\epsilon/dz << 1$. This motivates a new and more general formulation for the filtered lifting line theory as presented in the following section.

4 | GENERALIZED FORMULATION

We start by writing the solution to the streamwise vorticity equation from the linearized form of the vorticity transport equation [1]

$$\omega_x(x,y,z) = \frac{1}{2U_\infty(z)\pi}\frac{\partial}{\partial z}\int_0^S \left[\text{erf}(\text{x}/\epsilon(\text{z}')) + 1\right]\frac{1}{\epsilon(z')^2}G(z')e^{-(y^2+(z-z')^2)/\epsilon(z')^2}dz'. \tag{2}$$

The vorticity is composed of the superposition of the curl of forces at every blade position. Quantities inside the integral (z') take into account the changes in ϵ along the span. Following Ref [1], we use Biot-Savart to obtain the velocity component normal to the inflow along the blade span:

$$u_y'(z) = \frac{1}{8\pi^2}\iiint \frac{z-z''}{(x''^2+y''^2+(z-z'')^2)^{3/2}}\frac{1}{U_\infty(z'')}\frac{\partial}{\partial z}\int_0^S \left[\text{erf}(\text{x}''/\epsilon(\text{z}')) + 1\right]\frac{G(z')}{\epsilon(z')^2}e^{-(y''^2+(z''-z')^2)/\epsilon(z')^2}dz'. \tag{3}$$

The original formulation in filtered lifting line theory assumes that the changes in $\epsilon(z')$ are small (i.e. $d\epsilon/dz << 1$), and moves them outside of the integral in z'. This enabled the use of integration by parts to obtain an expression involving dG/dz (see Equation 1). To take into account the

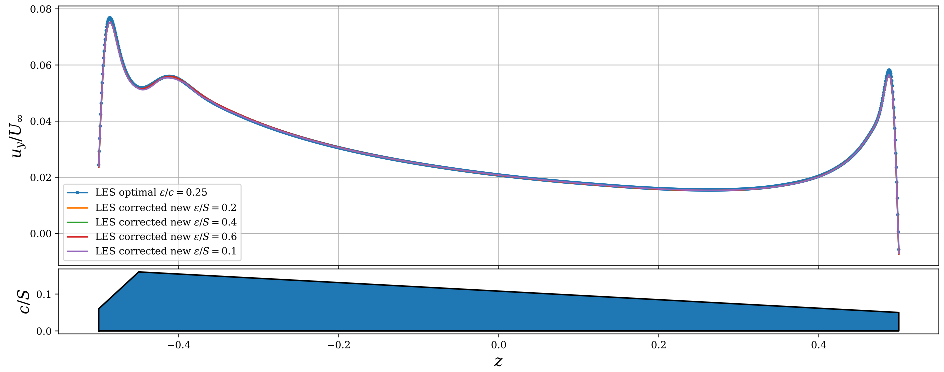

Figure 2 The induced velocity from the LES using the filtered lifting line theory correction with the newly developed formulation match results with optimal ϵ without the correction.

variations of ϵ along the blade, all terms should instead be kept inside the z' integration. Note that while LES may use a z-independent ϵ_{LES}, the correction also depends on $\epsilon_{\mathrm{opt}} = 0.25c$, which is a function of z if c varies along the span. Evaluating the integrals analytically (aided by the use of integral tables and Mathematica software [12,13,14]) leads to the new formulation for induced velocity in filtered lifting line theory:

$$u_y'(z) = -\frac{1}{2\pi}\int\limits_0^S \frac{1}{U_\infty(z')}\frac{G(z')}{\epsilon(z')^2}\left[e^{-(z'-z)^2/\epsilon(z')^2} + \frac{\epsilon(z')^2}{2(z'-z)^2}\left(e^{-(z'-z)^2/\epsilon(z')^2} - 1\right)\right]dz'. \quad (4)$$

Equation 4 is the generalized form of the filtered lifting line theory equations which is applicable to blades with significant changes in chord along the span, such as wind turbine blades. Note that unlike Equation 1 which is an integro-differential equation, Equation 4 is an integral equation because it involves G instead of dG/dz. It is worth highligthing that in the limit of $z \to z'$ the function inside the integral tends to $G(z')/2(U_\infty(z')\epsilon(z')^2)$, i.e. it does not diverge.

5 | IMPROVEMENTS

We use the new formulation and implement the filtered lifting line theory correction in our LES solver using Equation 4. We perform the same simulations from section 3 using the new formulation and compare the results to the simulation with the optimal ϵ. Figure 2 shows the induced velocity for simulations with the corrections compared to the high-fidelity simulation without the correction. There is excellent agreement within all the simulations, with the mean difference in induced velocity being withing less that 1% of the simulation using $\epsilon_{\mathrm{opt}} = 0.25c$. These results confirm that the new generalized formulation for filtered lifting line theory can capture changes along the span with high accuracy.

6 | CONCLUSIONS

The filtered lifting line theory was developed to predict the induced velocity and forces along a wing modeled by Gaussian body forces. The original formulation made assumptions that limited the accuracy of the theory when applied to blades that have significant changes in chord along the span, such as wind turbine blades. We reformulate filtered lifting line theory and obtain a new and more general formulation that can be applied to wings that have strong variations of chord along the blade span. We test the new formulation when using the filtered lifting line theory correction and obtain excellent agreement with high-fidelity simulations with optimal $\epsilon_{\mathrm{opt}}(z) = 0.25c(z)$.

This note improves the original formulation of the filtered lifting line theory by relaxing its assumptions. This work represents a step towards an even more generalized filtered lifting line theory. Future work should focus on a formulation that covers swept wings and includes drag force.

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Author contributions

LAMT reformulated the filtered lifting line theory equations and performed simulations. PS and LAMT implemented the software. CM and MC provided feedback and helped with the derivation.

Financial disclosure

None reported.

Conflict of interest

The authors declare no potential conflict of interests.

□

APPENDIX

A ORIGINAL LIFTING LINE THEORY

In the limit of $\epsilon \to 0$, the filtered lifting line theory equations are reduced to:

$$u_y'(z) = \frac{1}{4\pi} \int_0^S \frac{1}{U_\infty(z')} \frac{G(z')}{(z'-z)^2} dz'$$

Using integration by parts, the integral reduces to the same formulas in the original lifting line theory, making both theories consistent in the limit of infinitesimal vorticity.

$$u_y'(z) = \frac{1}{4\pi} \int_0^S \frac{\partial G(z')}{\partial z'} \frac{1}{U_\infty(z')(z'-z)} dz'$$

B FILTERED LIFTING LINE THEORY CORRECTION

The induced velocity from filtered lifting line theory can be used as a subgrid-scale model to correct the sampled velocity of the actuator line model[1]. At every time-step, the induced velocity is computed by using the information from the previous time-step. To this end, the correction can be derived by taking the difference between the induced velocity from a simulation with ϵ_{LES} and from using ϵ_{opt}. The following algorithm describes how to compute the induced velocity correction at every time-step using the information from the previous time-step for the proposed generalized filtered lifting line theory.

Algorithm 1 Pseudocode for the generalized filtered lifting line theory correction

for every z **do**
  Sample the velocity and add the correction from the previous time $u = u_{\text{LES}} + \Delta u$
  Using u, compute the angle of attack $\alpha(z)$, $c_l(z)$, and $W(z)$
  Compute $G(z) = \frac{1}{2} c_L(z) c(z) W^2(z)$
end for
for every z **do**
  Compute numerically $u_{\text{les}}(z) = \frac{1}{2\pi} \int \frac{1}{U_\infty(z')} \frac{G(z')}{\epsilon_{\text{les}}(z')^2} \left[e^{-(z'-z)^2/\epsilon_{\text{les}}(z')^2} + \frac{\epsilon_{\text{les}}(z')^2}{2(z'-z)^2} \left(e^{-(z'-z)^2/\epsilon_{\text{les}}(z')^2} - 1 \right) \right] dz'$
  Compute numerically $u_{\text{opt}}(z) = \frac{1}{2\pi} \int \frac{1}{U_\infty(z')} \frac{G(z')}{\epsilon_{\text{opt}}(z')^2} \left[e^{-(z'-z)^2/\epsilon_{\text{opt}}(z')^2} + \frac{\epsilon_{\text{opt}}(z')^2}{2(z'-z)^2} \left(e^{-(z'-z)^2/\epsilon_{\text{opt}}(z')^2} - 1 \right) \right] dz'$
  $\Delta u(z) = u_{\text{opt}}(z) - u_{\text{les}}(z)$
end for